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Exact Solutions for the Fluctuations in a Flat FRW Universe Coupled to a Scalar Field

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Some exact solutions for the small-first-order perturbations of an FRW metric minimally coupled to a neutral massive scalar field are presented.

KEY WORDS: massless scalar field; FRW; fluctuation.

1. INTRODUCTION

Since the early days of general relativity exact solutions have been extremely important in the development of this theory. Besides, in the last decade the study of fluctuations of homogeneous cosmological models has become utmost important for the global comprehension of the universe. In this paper, coupling both ideas, we begin the search of an exact solution for the equations of these fluctuations.

This paper is organized as follows.

In Section 2, we introduce our model: fluctuation in a flat FRW universe with n minimally coupled massless scalar fields with constant potential (namely a cosmological constant). We use the equations from the papers by Mukhanov *et al.* (1992), Zibin *et al.* (2001), and Greene *et al.* (1997).

In Section 3, we develop our calculations and find the solution for the background equations for the case $V = V_0 = \text{const.}$

In Section 4, we consider the fluctuation equations and find a new way to write these equations. Then in Subsection 4.1 we find an exact equation for the case $k = 0$, and in Section 4.2 we consider the case $k \neq 0$ and find an exact equation for the case $V_0 = 0$ and asymptotic solutions for the case $V_0 \neq 0$.

In Section 5, we draw our main conclusion.

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2. THE MODEL

Our metric is the flat ($K = 0$) FRW metric:

$$ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2), \tag{1}$$

where t is the proper time and $a(t)$ is the scalar factor.

The Lagrangian density of the system is $\mathcal{L}$, which corresponds to n neutral massless scalar fields ϕ_i, minimally coupled, with potential $V(\phi_1, \ldots, \phi_n)$:

$$\mathcal{L} = \mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{M}}, \tag{2}$$

where $\mathcal{L}_{\mathrm{G}} = -\frac{1}{12} R$ is the gravitational Lagrangian density (R is the Ricci scalar) and

$$\mathcal{L}_{\mathrm{M}} = \sum_{i=1}^{n} \left(-\frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{12} R \phi_i^2 \right) + V(\psi_1, \ldots, \psi_n) \tag{3}$$

is the matter Lagrangian density. The Ricci scalar is related to the scale factor a by the equation

$$\frac{R}{6} = \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2. \tag{4}$$

The background equations are the Klein–Gordon equations for each field:

$$\ddot{\phi}_i + 3H\dot{\phi}_i + \frac{\partial V}{\partial \phi_i} = 0, \quad i = 1, \ldots, n, \tag{5}$$

where $H = \dot{a}/a$, and the Hamiltonian constraint:

$$\mathcal{H}^2 = \frac{8\pi}{3m_{pl}^2} \left(V + \sum_{i=1}^{n} \frac{1}{2} \dot{\phi}_i^2 \right), \tag{6}$$

The perturbed metric reads

$$ds^2 = (1 - 2\Phi)\, dt^2 - a(t)^2 (1 + 2\Phi)(dx^2 + dy^2 + dz^2), \tag{7}$$

while the field perturbations of ϕ_i are symbolized as $\delta\phi_i$. Then, the k Fourier transform of the equations for the perturbations is given in Mukhanov *et al.* (1992) and Zibin *et al.* (2001):

$$3H\dot{\Phi} + \left(\frac{k^2}{a^2} + 3H^2 \right) = -\frac{3\lambda}{2} \sum_{i=1}^{n} \left(\dot{\phi}_i \delta\dot{\phi}_i - \Phi \dot{\phi}_i^2 + \frac{\partial V}{\partial \phi_i} \delta\phi_i \right), \tag{8}$$

$$\dot{\Phi} + H\Phi = \frac{3\lambda}{2} \sum_{i=1}^{n} \dot{\phi}_i \delta\phi_i, \tag{9}$$

$$\delta\ddot{\phi}_i + 3H\delta\dot{\phi}_i + \sum_{j=1}^{n}\frac{\partial V}{\partial\phi_i\partial\phi_j}\delta\phi_j = 4\dot{\Phi}\dot{\phi}_i - 2\frac{\partial V}{\partial\phi_i}\Phi - \frac{k^2}{a^2}\delta\phi_i, \quad i = 1, 2, \ldots, n, \tag{10}$$

where $\cdot = d(\ )/dt$ and $\lambda = 8\pi/3m_{pl}^2$.

3. THE EXACT SOLUTION FOR *V*, A CONSTANT

In this paper we will only be interested in the study of the case $V = V_0$ with $V_0 \geq 0$ (Cornish and Levin, 1996; Easther and Maeda, 1999). In this case, Eq. (5) becomes

$$\ddot{\phi}_i + 3H\dot{\phi}_i = 0, \quad i = 1, 2, \ldots, n, \tag{11}$$

$$H^2 = \lambda\left(V_0 + \frac{1}{2}\sum_{i=1}^{n}\dot{\phi}_i^2\right), \tag{12}$$

from which we deduce $\ddot{\phi}_i/\dot{\phi}_i = \ddot{\phi}_1/\dot{\phi}_1$; therefore,

$$\phi_i = c_i\phi_i + d_i, \quad i = 2, 3, \ldots, n \tag{13}$$

with c_i, d_i being constants. Using Eqs. (12) and (13) we obtain

$$H^2 = \lambda\left(V_0 + c^2\dot{\phi}_1^2\right), \tag{14}$$

where $c^2 = (1 + \sum_{i=2}^{n} c_i^2)/2$.

Taking (11) and (14) into account, we write

$$\sqrt{\lambda} = \mp\frac{1}{3}\frac{\ddot{\phi}_1}{\dot{\phi}_1\sqrt{c^2\dot{\phi}_1^2 + V_0}}. \tag{15}$$

On the other hand, since $H = \dot{a}/a$ and by Eq. (11) for $i = 1$, we have

$$\dot{\phi}_1 = \frac{c_1}{a^3}, \tag{16}$$

with c_1 being a certain constant.

3.1. Case $V_0 \neq 0$

In this case, the scale factor is obtained from Eqs. (15) and (16):

$$a^3(t) = \frac{c_1}{2V_0}(e^{-\Omega} - c^2 V_0\, e^{\Omega}), \tag{17}$$

with $\Omega = \mp 3\sqrt{\lambda V_0}(t - t_0)$.

Since $c > 0$ and we have assumed that $V_0 > 0$, we may consider the constant $\gamma = \ln c + \frac{1}{2} \ln V_0$; so,

$$a^3(t) = -\frac{c_1 c}{\sqrt{V_0}} \sinh(\Omega + \gamma).$$

Using Eqs. (16) and (17) and as we have assumed that $V_0 > 0$, after some algebraic calculations we deduce that

$$\dot{\phi}_1 = \frac{2V_0\, e^{\Omega}}{1 - c^2 V_0\, e^{2\Omega}} \tag{18}$$

and

$$\phi_1(t) = \mp\frac{1}{3c\sqrt{\lambda}} \ln\left(\frac{1 + c\sqrt{V_0}}{1 - c\sqrt{V_0}}\right) + \alpha, \tag{19}$$

where α is an integration constant.

3.2. Case $V_0 = 0$

In this case, the scale factor is obtained from Eqs. (15) and (16):

$$a^3(t) = \pm\, 3c_1\, c\sqrt{\lambda}(t - t_0).$$

From Eqs. (16) and (17) we get

$$\phi_1(t) = \pm\frac{1}{3c\sqrt{\lambda}} \ln(t - t_0) + \alpha,$$

where α is a constant.

4. THE PERTURBED SYSTEM SOLUTION

In the case $V = V_0$, following the work of Mukhanov *et al.* (1992), we use Eqs. (8), (9), and (10) to get

$$\delta\ddot{\phi}_i + 3H\delta\dot{\phi}_i = 4\dot{\Phi}\dot{\phi}_i - \frac{k^2}{a^2}\delta\phi_i, \quad i = 1, 2, \ldots, n, \tag{20}$$

$$\dot{\Phi} + H\Phi = \frac{3\lambda}{2}\sum_{i=1}^{n} \dot{\phi}_i\delta\phi_i, \tag{21}$$

$$3H\dot{\Phi} + \left(\frac{k^2}{a^2} + 3H^2\right) = -\frac{3\lambda}{2}\sum_{i=1}^{n} \left(\dot{\phi}_i\delta\dot{\phi}_i - \Phi\dot{\phi}_i^2\right), \tag{22}$$

where $k = 2\pi/l$, l being the wavelength. From Eqs. (13) and (16), we deduce

$$\dot{\phi}_i = \frac{k_i}{a^3}, \quad i = 1, 2, \ldots, n, \tag{23}$$

with k_i constants being $k_1 = c_1$ and $k_i = c_1 c_i, i = 2, 3, \ldots, n$. Therefore Eqs. (20), (21), and (22) become

$$\delta\ddot{\phi}_i + \frac{3\dot{a}}{a}\delta\dot{\phi}_i = 4\dot{\Phi}\frac{k_i}{a^3} - \frac{k^2}{a^2}\delta\phi_i, \quad i = 1, 2, \ldots, n, \tag{24}$$

$$\dot{\Phi} + \frac{\dot{a}}{a}\Phi = \frac{3\lambda}{2}\sum_{i=1}^{n}\frac{k_i}{a^3}\delta\phi_i, \tag{25}$$

$$3\frac{\dot{a}}{a}\dot{\Phi} + \left(\frac{k^2}{a^2} + 3H^2\right) = -\frac{3\lambda}{2}\sum_{i=1}^{n}\left(\frac{k_i}{a^3}\delta\dot{\phi}_i - \frac{k_i^2}{a^6}\Phi\right). \tag{26}$$

From Eqs. (12) and (23) we obtain

$$H = \pm\sqrt{\lambda}\sqrt{\frac{A^2}{a^6} + V_0}, \tag{27}$$

where $A^2 = c_1^2 c^2$ and c is the same as before.

Taking Eq. (27) into account we get

$$\dot{a} = \pm\sqrt{\lambda}\frac{\sqrt{A^2 + V_0 a^6}}{a^2}$$

and

$$\ddot{a} = \lambda\frac{-2A^2 + V_0 a^6}{a^5}.$$

Since $a(t)$ is a monotonic function, we consider the change $t \to z$ to the new independent variable $z = a(t)$, and writing $' = d/dz$, we have Eqs. (24), (25), and (26):

$$z^3\delta\phi_i'' + 3\sqrt{\lambda(A^2 + V_0 z^6)}\delta\phi_i' = 4k_i\Phi' - k^2 z\delta\phi_i, \quad i = 1, 2, \ldots, n, \tag{28}$$

$$\sqrt{\lambda(A^2 + V_0 z^6)}\Phi + z^3\Phi' = \frac{3\lambda}{2}\sum_{i=1}^{n}k_i\delta\phi_i, \tag{29}$$

$$3\sqrt{\lambda(A^2 + V_0 z^6)}z^3\Phi' + 3\lambda V_0 z^6 + k^2 z^4 + 3A^2\lambda = \frac{3\lambda}{2}\sum_{i=1}^{n}k_i^2\Phi - k_i z^3\delta\phi_i. \tag{30}$$

Let $S(z) = \sum_{i=1}^{n} k_i\delta\phi_i$. Multiplying each of the n equations of (28) by k_i and adding all of them, we get

$$z^3 S'' + 3\sqrt{\lambda(A^2 + V_0 z^6)}S' = 8A^2\Phi' - k^2 z S, \tag{31}$$

$$\sqrt{\lambda(A^2 + V_0 z^6)}\Phi + z^3\Phi' = \tfrac{3\lambda}{2}S, \tag{32}$$

$$3\sqrt{\lambda(A^2 + V_0 z^6)}z^3\Phi' + 3\lambda V_0 z^6 + k^2 z^4 + 3A^2\lambda = \tfrac{3\lambda}{2}(z^3 S' + 2A^2\Phi). \tag{33}$$

Now we will perform two steps.

First, equating $\Phi(z)$ from (31) and (32) we get

$$\begin{aligned}&\lambda z^2(A^2+V_0z^6)^2S'''(z)+2\lambda z(A^2+V_0z^6)(2A^2+5V_0z^6)S''(z)\\&+\big(-10A^4\lambda+A^2k^2z^4+19A^2\lambda V_0z^6+k^2V_0z^{10}+20\lambda V_0^2z^{12}\big)S'(z)\\&+z^3(5A^2k^2+36A^2\lambda V_0z^2+2k^2V_0z^6)S(z)=0.\end{aligned}\tag{34}$$

We consider the functional change $S(z)\to p(x)$ being $S(x)=\sqrt{a^2+V_0z^6}\,p(z)$, and we have

$$\begin{aligned}U:=\;&\lambda z^2(A^2+V_0z^6)p'''(z)+\lambda z(4A^2+19V_0z^6)p''(z)\\&+(-10A^2\lambda+k^2z^4+98\lambda V_0z^6)p'(z)+z^3(5k^2+126\lambda V_0z^2)p(z)=0.\end{aligned}\tag{35}$$

Second, equating $\Phi(z)$ from (32) and (33) we get

$$\begin{aligned}&\lambda z(-3A^2\lambda+k^2z^4)(A^2+V_0z^6)S''(z)\\&+\lambda(-15A^4\lambda+A^2k^2z^4-24A^2\lambda V_0z^6+4k^2V_0z^{10})S'(z)\\&+z^3(-15A^2k^2\lambda-36A^2\lambda^2V_0z^2+k^4z^4)S(z)=0.\end{aligned}\tag{36}$$

By the functional change $S(z)=\sqrt{a^2+V_0z^6}\,p(z)$ we have

$$\begin{aligned}W:=\;&\lambda z(-3A^2\lambda+k^2z^4)(A^2+V_0z^6)p''(z)\\&+\lambda(-15A^4\lambda+A^2k^2z^4-42A^2\lambda V_0z^6+10k^2V_0z^{10})p'(z)\\&+z^3(-15A^2k^2\lambda-126A^2\lambda^2V_0z^2+k^4z^4+18k^2\lambda V_0z^6)p(z)=0.\end{aligned}\tag{37}$$

From Eqs. (36) and (37) we deduce $U=1/k^2z^4-3\lambda A^2(z\frac{d}{dx}W-2W)$, and we conclude that if $W=0$, then all Eqs. (20), (21), and (22) are verified. From now on we will use Eq (35) with the U just defined and the last equation with $W=0$. With these changes the last three equations will be our system of equations.

4.1. Case $k=0$

We want to solve Eq. (36) for the case $k=0$; so

$$z(A^2+V_0z^6)S''(z)+(5A^2+8V_0z^6)S'(z)+12V_0z^5S(z)=0.\tag{38}$$

We see that z^{-4} is a particular solution for (38). Let us consider the functional change $S(z)=q(z)z^{-4}$ in (38) and we get

$$z(A^2+V_0z^6)q''(z)-3A^2q'(z)=0.$$

Integrating,

$$q'(z)=\frac{C_1z^3}{\sqrt{A^2+V_0z^6}},\qquad q(z)=C_1\int_{s_0}^{z}\frac{s^3}{\sqrt{A^2+V_0s^6}}\,ds+C_2,$$

where C_1 and C_2 are arbitrary constants. Therefore,

$$S(z) = \frac{1}{z^4}\left(C_1 \int_{s_0}^{z} \frac{s^3}{\sqrt{A^2 + V_0 s^6}}\, ds + C_2\right)$$

and

$$\Phi(z) = -\frac{\sqrt{\lambda}}{2A^2}\sqrt{A^2 + V_0 z^6}\, S(z) + C_1 \frac{\sqrt{\lambda}}{2A^2}.$$

We recall the change on the independent variable already made:

$$z^3 = a^3(t) = \frac{c}{2V_0}(e^{-\Omega} - c^2 V_0\, e^{\Omega}),$$

And since $e^{-\Omega} + c^2 V_0\, e^{\Omega} = 2c\sqrt{V_0}\, \cosh(\Omega_1)$ and $e^{-\Omega} - c^2 V_0 e^{\Omega} = -2c\sqrt{V_0}$ $\sinh(\Omega_1)$, we get

$$\Phi(t) = \frac{\sqrt{\lambda}}{2c^2 c_1^2}\left[C_1\left(1 - \frac{2c\sqrt{V_0}\, \cosh(\Omega_1)}{3(2c\sqrt{V_0}\, \sinh(\Omega_1))^{4/3}} \int_{\omega_0}^{\Omega_1} (2c\sqrt{V_0}\, \sinh(\omega))^{1/3} d\omega\right)\right.$$
$$\left. \times\, C_4 \frac{2c\sqrt{V_0}\, \cosh(\Omega_1)}{3(2c\sqrt{V_0}\, \sinh(\Omega_1))^{4/3}}\right], \tag{39}$$

where $\Omega_1 = \Omega + \gamma = \mp 3\sqrt{\lambda V_0}(t - t_1)$.

4.2. Case $k \neq 0$

We want to solve Eqs. (31), (32), and (33), and we equate $S(z)$ from (32). We have

$$S(z) = \frac{\sqrt{A^2 + V_0 z^6}\,(2\Phi + 2z\Phi')}{3\sqrt{\lambda}}.$$

We substitute this value in (31) and (33) and we obtain

$$\begin{aligned} U_k := {} & (2\sqrt{A^2 + V_0 z^6}(k^2 z^3 \Phi(z) + 18\lambda V_0 z^5 \Phi(z) - 10A^2 \lambda \Phi'(z) + k^2 z^4 \Phi'(z) \\ & + 38\lambda V_0 z^6 \Phi'(z) + 4A^2 \lambda z \Phi''(z) + 13\lambda V_0 z^7 \Phi''(z) + A^2 \lambda z^2 \Phi'''(z) \\ & + \lambda V_0 z^8 \Phi'''(z))/(3\sqrt{\lambda} z^5) = 0, \\ W_k := {} & (k^2 z^3 \Phi(z) + 6\lambda V_0 z^5 \Phi(z) + 5A^2 \lambda \Phi'(z) + 8\lambda V_0 z^6 \Phi'(z) \\ & + A^2 \lambda z \Phi''(z) + \lambda V_0 z^7 \Phi''(z))(z^5) = 0. \end{aligned}$$

It is easy to see that

$$U_k = \frac{2\sqrt{A^2 + V_0 z^6}}{\sqrt{\lambda}}\left[W_k + \frac{z}{3}\frac{d}{dz} W_k\right].$$

Therefore, the only equation to be solved is $W_k = 0$. We first consider the case when $V_0 = 0$ and afterwards when $V_0 \neq 0$.

If $V_0 = 0$, we have

$$A^2 \lambda z \Phi''(z) + 5A^2 \lambda \Phi'(z) + k^2 z^3 \Phi = 0. \tag{40}$$

We write $\bar{k} = k/(A\sqrt{\lambda})$. We consider the change of variable $z = \sqrt{2x/\bar{k}}$ and the functional change $G(x) = x\Phi(x)$ and the equation obtained is

$$x^2 \frac{d^2 G}{dx^2} + x \frac{dG}{dx} + (x^2 - 1)G = 0. \tag{41}$$

The exact solution of Eq. (41) is $G(x) = C_J J_1(x) + C_Y Y_1(x)$, where C_J and C_Y are arbitrary constants and J_1 and Y_1 are the corresponding Bessel functions with index 1. [See Abramowitz and Stegun (1965) for further information about Bessel functions.] Going back through the changes, we deduce

$$\Phi(z) = \frac{1}{\eta} \left(C_J J_1(\eta) + C_Y Y_1(\eta) \right), \tag{42}$$

where $\eta = kz^2/2A\sqrt{\lambda}$.

If $V_0 \neq 0$, we have $W_k = 0$, and by the changes $z = \sqrt{A^{2/3} V_0^{-1/3} x}$ and $\bar{k}^{-3} = k^{-3} A V_0 \lambda^{3/2}$ we get

$$4x(1 + x^3)\Phi''(x) + 6(2 + 3x^3)\Phi'(x) + (\bar{k}^2 x + 6x^2)\Phi(x) = 0. \tag{43}$$

If $x \gg 1$, then

$$\Phi(x) \simeq \frac{c_1}{x^3} + \frac{c_2}{\sqrt{x}},$$

If $x \gg 1$, then

$$\Phi(x) \simeq \frac{i}{8kx} \left(c_1 K_1 \left(\frac{1}{2} kxi \right) + c_2 I_1 \left(\frac{1}{2} kxi \right) \right),$$

where $i^2 = -1$ and c_1, c_2 are constants.

5. CONCLUSIONS

We have found exact solutions for the cases $k = 0$ and $k \neq 0$, $V_0 = 0$, and asymptotic exact solutions for the case $k \neq 0$, $V_0 \neq 0$. Then the exact solution for the general case $k \neq 0$, $V_0 \neq 0$ is not very far and will be treated elsewhere. There we will face more general potentials in trying to find exact solutions or perturbing these exact solutions to shed more light on the fluctuation problems.

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